JUSTIFICATION OF THE INTEGRAL CRITERION OF STABILITY OF MOTION IN PROBLEMS ON AUTOSYNCHRONIZATION OF VIBRATORS

(OBOSNOVANIE INTEGRAL'NOGO PRIZNAKA USTOICHIVOSTI DVIZHENIIA V ZADACHAKH O SAMOSINKHRONIZATSII VIBRATOROV)

PMM Vol.24, No.6, 1960, pp. 1100-1103

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(Received April 6, 1960)

The validity of the integral criterion of stability of synchronized motion in problems on autosynchronization of mechanical vibrators was first established for a number of special examples [1]. An investigation of those same examples suggested the idea of the possibility of simplifying the derivation of the relations for the determination of the phases of rotation of the vibrators in synchronized motions.

In this paper it is shown that, in the general case, the results of the investigation of synchronized motions obtained in using a simplified method for the determination of the phases and of the integral criterion of stability of motion coincide exactly with the results of the solution of the problem by the methods of Poincaré and Liapunov [2].

1. Let us consider the more general case of the problem on the autosynchronization of an arbitrary number k of mechanical vibrators when the latter are installed on one or several solid bodies (vibrating organs) which are connected with each other and with a fixed base by means of a system of elastic elements having ν degrees of freedom. The deviations of the mentioned bodies from the position of static equilibrium are described by generalized coordinates $x_1, \ldots, x_\nu$, while the position of the rotors of the vibrators is given in terms of their angles of rotation $\phi_1, \ldots, \phi_k$ relative to some fixed direction.

Synchronized motions of a system are motions of the form

$$\varphi_s = \sigma_s[\omega t + \psi_s(\omega t)] \quad (s = 1, \ldots, k), \qquad x_r = x_r(\omega t) \quad (r = 1, \ldots, \nu) \qquad (1.1)$$

where ψ_s and x_r are periodic functions of time with period $2\pi/\omega$, $\sigma_s = \pm 1$.

In the treatment of the problem on synchronized motions of vibrators with equal and positive partial velocities far away from resonance by the methods of Poincaré and Liapunov [2,3], the initial approximation (generating solution)

$$\varphi_s^{(0)} = \sigma_s(\omega t + \alpha_s), \qquad x_r^{(0)} = x_r^{(0)}(\omega t) \tag{1.2}$$

is easily seen to satisfy the equations*

$$\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_r} - \frac{\partial L}{\partial x_r}\right]_0 = 0 \qquad (r = 1,\ldots,\nu) \tag{1.3}$$

$$\sigma_s \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}_s} - \frac{\partial L}{\partial \varphi_s}\right]_0 dt \equiv W_s(\alpha_1,\ldots,\alpha_k) = 0 \qquad (s = 1,\ldots,k) \tag{1.4}$$

where $L = L(x_1, \ldots, x_\nu; \dot{x}_1, \ldots, \dot{x}_\nu; \phi_1, \ldots, \phi_k, \dot{\phi}_1, \ldots, \dot{\phi}_k)$ are functions of the Lagrange system, while the W_s are quantities which are called in [3] vibrational moments; L is a periodic function of ϕ_3 of period 2π. From the last k equations one can determine (to within an additive constant) the values of the "generating phases" $a_s = a_s^*$ to which there can correspond synchronized motions.

2. Before we pass to the proof of the proposition of [1], we shall prove a more general assertion which can be used with slight modifications also in the solution of other nonlinear problems.

We shall show that Equations (1.4) for the determination of the generating phases a_s^* coincide with the conditions for the stability of the mean value over the period $2\pi/\omega$ of Lagrange's function of the entire system evaluated for the generating solution (1.2)

$$\Lambda = \Lambda(\alpha_1,\ldots,\alpha_k) = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}[L]_0\,dt \tag{2.1}$$

and that the conditions of stability of synchronized motions, found by the methods of Poincaré and Liapunov [2,3] for the case of vibrators which have the same positive partial velocities, can be obtained from the requirement that the values of the phases a_s^*, which correspond to the synchronized motion under consideration, should make the function $\Lambda(a_1, \ldots, a_k)$ a maximum.

Let us compute the derivative $\partial\Lambda/\partial a_s$. After some simple operations which involve integration by parts, the use of the relations (1.3), (1.4)

* The square brackets with the subscript $_0$ denote that the enclosed functions of the generalized coordinates are evaluated for the generating solution (1.2).

and a consideration of the periodicity of the solution (1.2), we obtain

$$\frac{\partial\Lambda}{\partial\alpha_s}=\frac{\omega}{2\pi}\sum_{j=1}^{\nu}\int_0^{2\pi/\omega}\left\{\left[\frac{\partial L}{\partial x_j}\right]_0\frac{\partial x_j^{(0)}}{\partial\alpha_s}+\left[\frac{\partial L}{\partial \dot{x}_j}\right]_0\frac{\partial \dot{x}_j^{(0)}}{\partial\alpha_s}\right\}dt+\frac{\omega}{2\pi}\sum_{j=1}^{k}\int_0^{2\pi/\omega}\left\{\left[\frac{\partial L}{\partial\varphi_j}\right]_0\frac{\partial\varphi_j^{(0)}}{\partial\alpha_s}+\right.$$

$$\left.+\left[\frac{\partial L}{\partial\dot{\varphi}_j}\right]_0\frac{\partial\dot{\varphi}_j^{(0)}}{\partial\alpha_s}\right\}dt=\frac{\omega}{2\pi}\sum_{j=1}^{k}\int_0^{2\pi/\omega}\left[\frac{\partial L}{\partial x_j}-\frac{d}{dt}\frac{\partial L}{\partial\dot{x}_j}\right]_0\frac{\partial x_j^{(0)}}{\partial\alpha_s}dt+\frac{\omega}{2\pi}\sum_{j=1}^{\nu}\int_0^{2\pi/\omega}\left[\frac{\partial L}{\partial\varphi_j}-\right.$$

$$\left.-\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}_j}\right]_0\frac{\partial\varphi_j^{(0)}}{\partial\alpha_s}dt=\sigma_s\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\left[\frac{\partial L}{\partial\varphi_s}-\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}_s}\right]_0 dt=-W_s(\alpha_1,\ldots,\alpha_k) \qquad (2.2)$$

Thus the condition $\partial\Lambda/\partial\alpha_s = 0$ coincides with Equations (1.4); and the first part of the proposition is proved.

For the proof of the second part we expand the function $\Lambda(\alpha_1, \ldots, \alpha_k)$ into a power series around the point which corresponds to the solution $\alpha_1^*, \ldots, \alpha_k^*$ of Equation (1.4). In view of (2.2) and (1.4) the linear term of this series is zero and

$$\Lambda(\alpha_1,\ldots,\alpha_k)-\Lambda(\alpha_1^*,\ldots,\alpha_k^*)=-\sum_{r=1}^{k}\sum_{j=1}^{k}\left(\frac{\partial W_r}{\partial\alpha}\right)_{\alpha=\alpha^*}(\alpha_r-\alpha_r^*)(\alpha_j-\alpha_j^*)+\ldots \qquad (2.3)$$

where the omitted terms are of order higher than two in the variables $\alpha_1-\alpha_1^*, \ldots, \alpha_k-\alpha_k^*$. In order that the function Λ has a maximum at the point $(\alpha_1^*, \ldots, \alpha_k^*)$, it is sufficient that the quadratic form

$$\Phi=-\sum_{r=1}^{k}\sum_{j=1}^{k}\left(\frac{\partial W_r}{\partial\alpha_j}\right)_{\alpha=\alpha^*}(\alpha_r-\alpha_r^*)(\alpha_j-\alpha_j^*) \qquad (2.4)$$

be definite-negative. In order that this may be true, it is necessary and sufficient [4] that all the roots of the algebraic kth degree equation (δ_{rj} in Kronecker's symbol)

$$\left|-\left(\frac{\partial W_r}{\partial\alpha_j}\right)_{\alpha=\alpha^*}-\delta_{rj}z\right|=0 \qquad (r,j=1,\ldots,k) \qquad (2.5)$$

be negative. We note that all the roots of Equation (2.5) are real because

$$\left(\frac{\partial W_r}{\partial\alpha_j}\right)_{\alpha=\alpha^*}=\left(\frac{\partial W_j}{\partial\alpha_r}\right)_{\alpha=\alpha^*}=-\left(\frac{\partial^2\Lambda}{\partial\alpha_r\,\partial\alpha_j}\right)_{\alpha=\alpha^*}$$

If even one of the roots of Equation (2.5) is positive, then the form (2.4) will no longer be a definite form, and there is no maximum. The case when there are zero roots is unsettled and requires, in general, an investigation of the terms of higher order in the expansion (2.3).

We note that in consequence of the autonomy of the initial system of equations, the constants α_s enter into the expressions for Λ and W_s only in the form of the differences $\alpha_r - \alpha_s$. Hence, the indicated expressions do not change if each α_r is replaced by an $\alpha_r + \alpha_0$, where α_0 is an arbitrary constant. The function Λ, therefore, does not change along the hyperline $\alpha_1 = \alpha_1^* + \alpha_0, \ldots, \alpha_k = \alpha_k^* + \alpha_0$, and Equation (2.5), in this case, has always one zero root. This, however, does not affect the argument on the nature of the stationary point. The requirement that the remaining k-1 roots of Equation (2.5) be negative is, therefore, a sufficient condition for a maximum of the function Λ. If, however, one of the roots is positive, a maximum cannot exist. The case of more than one zero root is not settled. These last results are, however, exactly the conditions for stability of synchronized motion obtained by the methods of Liapunov and Poincaré [2,3].

This completes the proof of the second part of the stated proposition.

Note. The presence of the root $z = 0$ in our case can be proved directly if one differentiates

$$W_r(\alpha_1^* + \alpha_0, \ldots, \alpha_k^* + \alpha_0) \equiv W_r(\alpha_1^*, \ldots, \alpha_k^*)$$

with respect to α_0 and then sets $\alpha_0 = 0$. One thus obtains the result that

$$\sum_{s=1}^{k} (\partial W_r/\partial \alpha_s)_{\alpha=\alpha^*} \equiv 0$$

from which it follows that if one adds to the elements of any one column of the determinant (2.5) the elements of all other columns then one obtains a column whose elements are equal to $-z$.

We note that if the stationary character of the function Λ for all possible synchronized motions follows from Hamilton's principle then, since Λ differs from the motion in the Hamilton sense for the corresponding system by a factor, the argument on the nature of the stationary point in the given case cannot be obtained by starting out with Hamilton's principle, for the motion is taken over a finite and not over a sufficiently small interval of time [5].

3. For the proof of the proposition given in [1] one needs to prove, on the basis of Equation (2.3), only the relation

$$\frac{\partial \Lambda}{\partial \alpha_s} = -\frac{\partial \Lambda_0}{\partial \alpha_s} = -W_s(\alpha_1, \ldots, \alpha_k) \tag{3.1}$$

where Λ_0 is the mean, over the period $2\pi/\omega$, of the value of Lagrange's function L_0 of the auxiliary bodies computed for the solution (1.2). Representing Lagrange's function L in the form

$$L = L_0 + L_1 \tag{3.2}$$

we note that L_0 is a quadratic form in $x_1, \ldots, x_\nu$ and $\dot{x}_1, \ldots, \dot{x}_\nu$, while L_1 is, in the given case, the sum of a quadratic form in the variables $\phi_1, \ldots, \phi_k$, of a linear form in $\dot{x}_1, \ldots, \dot{x}_\nu$ with coefficients depending on $\phi_1, \ldots, \phi_k$ and $\dot{\phi}_1, \ldots, \dot{\phi}_k$, and of a periodic (with period 2π) function (depending only on the coordinates $\phi_1, \ldots, \phi_k$) whose mean value over the period is zero.

In view of what has been said, we have

$$\int_0^{2\pi/\omega} \sum_{j=1}^{\nu} \dot{x}_j^{(0)} \left[\frac{\partial L_1}{\partial \dot{x}_j}\right]_0 dt = \int_0^{2\pi/\omega} [L_1]_0\, dt + B, \qquad \sum_{j=1}^{\nu} \left(\dot{x}_j \frac{\partial L_0}{\partial \dot{x}_j} + x_j \frac{\partial L_0}{\partial x_j}\right) = 2L_0 \tag{3.3}$$

where B is a quantity independent of $\alpha_1, \ldots, \alpha_k$. Differentiating these identities, which were set up for the solution (1.2) with respect to α_s, we obtain after some simplifications

$$\int_0^{2\pi/\omega} \sum_{j=1}^{\nu} \dot{x}_j^{(0)} \frac{\partial}{\partial \alpha_s}\left[\frac{\partial L_1}{\partial \dot{x}_j}\right]_0 dt = \int_0^{2\pi/\omega} \sum_{j=1}^{k} \left\{\left[\frac{\partial L_1}{\partial \varphi_j}\right]_0 \frac{\partial \varphi_j^{(0)}}{\partial \alpha_s} + \left[\frac{\partial L_1}{\partial \dot{\varphi}_j}\right]_0 \frac{\partial \dot{\varphi}_j^{(0)}}{\partial \alpha_s}\right\} dt \tag{3.4}$$

$$\sum_{j=1}^{\nu} \left\{ x_j^{(0)} \frac{\partial}{\partial \alpha_s}\left[\frac{\partial L_0}{\partial x_j}\right]_0 + \dot{x}_j^{(0)} \frac{\partial}{\partial \alpha_s}\left[\frac{\partial L_0}{\partial \dot{x}_j}\right]_0\right\} = \sum_{j=1}^{\nu} \left\{\left[\frac{\partial L_0}{\partial x_j}\right]_0 \frac{\partial x_j^{(0)}}{\partial \alpha_s} + \left[\frac{\partial L_0}{\partial \dot{x}_j}\right]_0 \frac{\partial \dot{x}_j^{(0)}}{\partial \alpha_s}\right\}$$

Making use of these equations and of (2.2), we find that

$$\frac{\partial \Lambda}{\partial \alpha_s} = \frac{\omega}{2\pi} \sum_{j=1}^{k} \int_0^{2\pi/\omega} \left\{\left[\frac{\partial L}{\partial \varphi_j}\right]_0 \frac{\partial \varphi_j^{(0)}}{\partial \alpha_s} + \left[\frac{\partial L}{\partial \dot{\varphi}_j}\right]_0 \frac{\partial \dot{\varphi}_j^{(0)}}{\partial \alpha_s}\right\} dt = \frac{\omega}{2\pi} \sum_{j=} \int \dot{x}_j^{(0)} \frac{\partial}{\partial \alpha_s}\left[\frac{\partial L_1}{\partial \dot{x}_j}\right]_0 dt$$

$$\frac{\partial \Lambda_0}{\partial \alpha_s} = \frac{\omega}{2\pi} \sum_{j=1}^{\nu} \int_0^{2\pi/\omega} \left\{\left[\frac{\partial L_0}{\partial x_j}\right]_0 \frac{\partial x_j^{(0)}}{\partial \alpha_s} + \left[\frac{\partial L_0}{\partial \dot{x}_j}\right]_0 \frac{\partial \dot{x}_j^{(0)}}{\partial \alpha_s}\right\} dt =$$

$$= \frac{\omega}{2\pi} \sum_{j=1}^{\nu} \int_0^{2\pi/\omega} \left\{ x_j^{(0)} \frac{\partial}{\partial \alpha_s}\left[\frac{\partial L_0}{\partial x_j}\right]_0 + \dot{x}_j^{(0)} \frac{\partial}{\partial \alpha_s}\left[\frac{\partial L_0}{\partial \dot{x}_j}\right]_0\right\} dt \tag{3.5}$$

Whence, bearing in mind (1.2) and (1.3), we obtain

$$\frac{\partial (\Lambda_0 + \Lambda)}{\partial \alpha_s} = \frac{\omega}{2\pi} \sum_{j=1}^{\nu} \int_0^{2\pi/\omega} \left\{ x_j^{(0)} \frac{\partial}{\partial \alpha_s}\left[\frac{\partial L_0}{\partial x_j}\right]_0 + \dot{x}_j^{(0)} \frac{\partial}{\partial \alpha_s}\left[\frac{\partial L}{\partial \dot{x}_j}\right]_0\right\} dt =$$

$$= \frac{\omega}{2\pi} \sum_{j=1}^{\nu} \int_0^{2\pi/\omega} \left\{ x_j^{(0)} \frac{\partial}{\partial \alpha_s}\left[\frac{\partial L}{\partial x_j}\right]_0 + \dot{x}_j^{(0)} \frac{\partial}{\partial \alpha_s}\left[\frac{\partial L}{\partial \dot{x}_j}\right]_0\right\} dt =$$

$$= \frac{\omega}{2\pi} \sum_{j=1}^{\nu} \int_0^{2\pi/\omega} \frac{d}{dt} \left\{ x_j^{(0)} \frac{\partial}{\partial \alpha_s}\left[\frac{\partial L}{\partial \dot{x}_j}\right]_0\right\} dt = 0 \tag{3.6}$$

This proves the validity of Equation (3.1). At the same time we have established the equivalence of the condition of stationary character of the function $\Lambda_0(a_1, \ldots, a_k)$ to the validity of Equation (1.4) for the determination of the generating phases $a_s = a_s^*$, and the equivalence of the condition of the existence of a minimum of this function to the condition of stability of the synchronized motion.

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Translated by H.P.T.